ELECTRIC AND THERMAL CONDUCTIVITY ANISOTROPY OF AN ARTICULATED MEDIUM

M. L. Kachanov

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It is well known that articulation may cause the electric and thermal conducting properties of a medium to become anisotropic. Extensive experimental data relating to the conductivity of rock were presented in [1], and according to these data the ambient resistance in the case of an orderly arrangement of cracks noticeably depends on the direction along which it is measured; an ellipsoid of resistance anisotropy can be constructed from the measurement results. A pronounced correlation between the orientations of the resistivity extrema and the orientation of the crack system is observed here. A comparison betwen the "direction rose" of the fractures and the "direction rose" of the articulation carried out for different regions has shown that they are identical. A crack density tensor T_α describing the average (with respect to a given volume) geometry of the articulation has been introduced [2, 3]. In the current work, it is proved that T_α can be effectively used in problems involving anisotropic electrical and thermal conductivity. The resistivity tensor σ and thermal-conductivity coefficient tensor K, which characterize the anisotropy of the electrical and thermal conducting properties, are expressed in terms of T_α. The structure of this relation is established; the equations presented allow us to find the form of σ and K if the articulation parameters are known.

Articulation geometry in a body containing N cracks [2, 3] is entirely described by the delta-shaped field of a bivalent symmetric tensor

$$T'_\alpha = \sum_{i=1}^{N} b_i \mathbf{n}_i \mathbf{n}_i \delta(S_i),$$

where b_i and $\mathbf{n}_i$ are the opening of the crack and the unit normal to its middle surface (in general, variables along each crack), $\delta(S_i)$ is a delta function concentrated on the surface S_i, and i is the number of the crack. The volume mean

$$\langle T'_\alpha \rangle_V = \frac{1}{V} \sum_{i=1}^{N} \int_{S_i(V)} b\mathbf{nn}dS \equiv T_\alpha \tag{1}$$

will be called the crack density tensor.

The crack density tensor is a purely geometric parameter, and must be somewhat modified for describing conducting properties. In fact (in the case of not very powerful electric fields), the contribution made by the i-th crack to an increase in the ambient resistance is independent of crack opening b_i, so that we will modify the crack density tensor T_α, defining it as the volume mean of the tensor field,

$$T''_\alpha = \sum_i \mathbf{n}_i \mathbf{n}_i \delta(S_i).$$

We find that

$$T_\alpha = \langle T''_\alpha \rangle_V = \frac{1}{V} \int_V \sum_i \mathbf{n}_i \mathbf{n}_i \delta(S_i)\, dV = \frac{1}{V} \sum_i \int_{S_i(V)} \mathbf{nn}dS \tag{2}$$

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is a bivalent symmetric tensor, giving an averaged (with respect to volume V) description of the articulation, while the contribution made by the i-th crack to the mean depends on the area and orientation of its middle surface and is independent of b_i. We note that the linear invariant of T_α is given by

$$SpT_\alpha = \frac{1}{V}\sum_i \int_{S_i(V)} (\mathbf{nn})\, dS = \frac{1}{2}\frac{\Sigma(V)}{V} \equiv \alpha,$$

where $\Sigma(V)$ is the total area of the surfaces of the banks of the cracks contained in volume V.

The dyad **nn** is taken outside the integration and summation signs in the inportant case of stratified articulation (a single system of parallel two-dimensional cracks) and

$$T_\alpha = \alpha\mathbf{nn}.$$

The tensor T_α is spherical in the case of "random" ("unsystematic") articulation.

It is well known that Ohm's law in an anisotropic medium has the form

$$\mathbf{j} = -\sigma \cdot \nabla\varphi, \tag{3}$$

where $\mathbf{j}$ is the current density vector, φ is electric field potential, and σ is the bivalent resistivity tensor (it is usually assumed to be symmetric in accordance with the Onsager principle [4]); the dot denotes convolution with respect to a single index. The values occurring in Eq. (3) are understood to be aveages with respect to some "elementary volume."

It is assumed that the medium is isotropic in terms of conducting properties, i.e., $\sigma = s_0 I$ (I is a unit tensor). We will also assume that articulation is the sole reason for the anisotropy. Then the difference $S_0 I - \sigma$, describing the variation of conductivity due to articulation, will be a function of the crack density tensor,

$$s_0 I - \sigma = f(T_\alpha). \tag{4}$$

Let us define this function concretely. We may see that f is an isotropic tensor function. In fact, this means, according to one definition [5], that

$$f(A \cdot T_\alpha \cdot A^*) = A \cdot f(T_\alpha) \cdot A^*, \tag{5}$$

where A is a tensor that defines an arbitrary linear orthogonal transformation and A* is the tensor conjugate to A. The meaning of Eq. (5) is that, as the systems of cracks undergoes "rotations" and "mirror reflections," the current vector will undergo a corresponding transformation. This condition holds if the material is itself isotropic. We may therefore use the Hamilton–Cayley theorem in order to represent Eq. (4) in the form

$$s_0 I - \sigma = c_0 I + c_1 T_\alpha + c_2 T_\alpha \cdot T_\alpha, \tag{6}$$

where the scalar coefficients c_0, c_1, and c_2 are functions of the invariants of T_α. Clearly, the tensors $s_0 I - \sigma$ and T_α arecoaxial.

Let us assume that current attenuation due to articulation is equal to the sum of the attenuations corresponding to each crack (weak interactions between cracks). That is, as the articulation in the elementary volume is (arbitrarily) divided into several groups of cracks,

$$T_\alpha = T_\alpha^{(1)} + T_\alpha^{(2)} + \dots$$

we will have

$$f(T_\alpha) = f(T_\alpha^{(1)}) + f(T_\alpha^{(2)}) + \dots .$$

It therefore follows that f is linear and homogeneous, i.e., Eq. (8) reduces to the form

$$s_0 I - \sigma = s_1 T_\alpha + s_2 (sp T_\alpha) I, \tag{7}$$

where s_1 and s_2 are scalar coefficients independent of T_α.

We now impose the natural requirement that the ambient conductivity in directions parallel to the cracks be the same in the case of stratified articulation as in the absence of cracks. (This requirement holds for cracks with a small opening that do not strongly influence the cross-sectional area perpendicular to them.) We direct the x_1 axis of the Cartesian coordinate system along the normal to the cracks **n**. From Eq. (7) we find that

$$\sigma = [s_0 - (s_1 + s_2)\alpha]e_1e_1 + [s_0 - s_2\alpha]e_2e_2 + [s_0 - s_2\alpha]e_3e_3$$

($\mathbf{e}_i$ is a unit vector along the x_i axis). It is therefore clear that this requirement leads to the condition $s_2 = 0$.

Thus, the resistivity tensor has the structure

$$\sigma = s_0 I - s_1 T_\alpha \tag{8}$$

or (along the principal axes),

$$\sigma = (s_0 - s_1\alpha_1)e_1e_1 + (s_0 - s_1\alpha_2)e_2e_2 + (s_0 - s_1\alpha_3)e_3e_3, \tag{9}$$

where α_i are the principal values of T_α that characterize the density of the articulation (area of free surfaces) in the $\mathbf{e}_i$ directions. Since articulation decreases ambient conductivity, $s_1 > 0$.

Let us consider the case of importance for rock in which the articulation is formed by several systems of parallel cracks. Here the vector $\mathbf{n}$ is constant within each system of cracks, so that removing the dyad $\mathbf{nn}$ from the integral signs in Eq. (2), we obtain

$$\sigma = s_0 I - s_1 \sum_i \mathbf{n}_i \mathbf{n}_i F_i, \tag{10}$$

where F_i, which characterizes the density of the articulation of the i-th system of cracks, is equal to the free surface area of the cracks, referred to an averaged volume V (in rock mechanics, F_i is sometimes referred to as the packed rock density [6]).

For a medium with stratified articulation $\alpha_2 = \alpha_3 = 0$, and

$$\sigma = (s_0 = s_1\alpha)e_1e_1 + s_0(e_2e_2 + e_3e_3), \tag{11}$$

so that it is clear that the ambient conductivity is identical in all directions parallel to the cracks.

In the case of random articulation ($\alpha_1 = \alpha_2 = \alpha_3$) the tensor is spherical and conductivity is isotropic. Equations (9)-(11) express σ in terms of the articulation parameters.

It was supposed above that the crack cavities were either empty or filled with a material which was assumed to have infinite resistance (air). In actual practice it is often the case that crack cavities are filled with a material of finite resistance. Such filling can be taken into account within the framework of our model in the following way.

If a filler is present, the opening b of the slits becomes a substantial factor, so that it is necessary to avoid modifying the tensor T_α and instead assume that it is determined by Eq. (1). We will take an effective value for the crack density tensor:

$$T_\alpha = \left(1 - \frac{s_0}{s_*}\right)\tilde{T}_\alpha,$$

where s_* is filler resistance and $\tilde{T}_\alpha$ is given by Eq. (1). When $s_* < s_0$, the components of T_α become negative. This corresponds to the natural fact that cracks containing a filler which constitutes a better conductor than the "basic" material decreases the ambient resistance. (A similar situation is typical for rock whose cracks are filled with different types of liquids.)

Heat transfer by thermal conductivity is analogous to the transfer of electrical energy. The basic equation will be the Fourier thermal-conductivity law $\mathbf{h} = -K \cdot \nabla\tau$, where $\mathbf{h}$ is the heat flow vector, τ is temperature, and K is a bivalent tensor of the thermal-conductivity coefficients which, as in the case of the resistivity tensor, is usually assumed to be symmetric. If the medium is isotropic in terms of conducting properties ($K = k_0 I$) in the absence of cracks, i.e., articulation is the sole cause of anisotropy, $K = K(T_\alpha)$.

Repeating the arguments used in defining the function (4) word for word, we arrive at the equation $K = k_0 I - k_1 T_\alpha$, which is analogous to Eq. (8).

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